How Accurate Is the Small-Angle Pendulum Formula? A Theoretical Analysis

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Abstract

The period of a simple pendulum is usually given as $T_0 = 2\pi\sqrt{L/g}$, which assumes the swing angle is small. This paper asks one question: how large is the error of this formula as the starting angle grows? Using the exact expression for the period in terms of the complete elliptic integral of the first kind, I compute the ratio between the exact and approximate periods for angles from 5 to 90 degrees. The error is below 1% up to about 20 degrees, reaches about 4% at 45 degrees, and is about 18% at 90 degrees. This is a calculation, not an experiment; I did not measure a real pendulum.

1 Introduction

The simple pendulum is one of the first systems studied in physics. A mass hangs from a light string of length L and swings under gravity with acceleration g. Textbooks give the period $T_0 = 2\pi\sqrt{L/g}$, which does not depend on the starting angle. This is only an approximation. My research question is: *for what range of angles is the approximation good enough?*

2 Method

For a starting angle θ_0, conservation of energy gives the exact period

$$T = 4\sqrt{\frac{L}{g}}\,K(k), \qquad k = \sin\frac{\theta_0}{2}, \tag{1}$$

where $K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$ is the complete elliptic integral of the first kind. Dividing by T_0 gives

$$\frac{T}{T_0} = \frac{2}{\pi}K(k). \tag{2}$$

I evaluated this ratio numerically with standard scientific software for several angles. Air resistance, string mass and friction at the pivot are ignored.

3 Results

Table 1 shows the ratio and the percentage by which the small-angle formula underestimates the true period.

Angle (degrees)	T/T_0	Error of T_0 (%)
5	1.0005	0.05
10	1.0019	0.19
20	1.0077	0.77
30	1.0174	1.74
45	1.0400	4.00
60	1.0732	7.32
90	1.1803	18.03

Table 1: Exact period divided by the small-angle period.

4 Discussion

The small-angle formula is accurate to within 1% for angles up to about 20 degrees, so it is reliable for many classroom experiments with small swings. At 45 degrees the period is about 4% longer than the formula predicts, and at 90 degrees about 18% longer. Larger swings therefore need the exact expression.

Limitations. This work is purely theoretical. It uses an ideal model, and a real experiment would also include measurement error and damping. A natural next step is to time a real pendulum at several angles and compare the results with Table 1.

5 Conclusion

The small-angle pendulum formula is a good approximation only for small swings. Beyond roughly 20 degrees the error becomes noticeable and grows quickly.

References

[1] J. B. Marion and S. T. Thornton, *Classical Dynamics of Particles and Systems*, Brooks/Cole, standard university textbook.

[2] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, chapter on elliptic integrals.